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Nonclassic boundary value problems in the theory of irregular systems of equations with partial derivatives¹

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Abstract. The linear PDE $\mathbf{B}\mathbf{L}(\frac{\partial}{\partial x})u = \mathbf{L}_1(\frac{\partial}{\partial x})u + f(x)$ with nonclassic conditions on boundary $\partial\Omega$ is considered. Here \mathbf{B} is linear noninvertible bounded operator acting from linear space E into E , $x = (t, x_1, \dots, x_m) \in \Omega$, $\Omega \subset \mathbb{R}^{m+1}$. It is assumed that \mathbf{B} enjoys the skeleton decomposition $\mathbf{B} = \mathbf{A}_1\mathbf{A}_2$, $\mathbf{A}_2 \in \mathcal{L}(E \rightarrow E_1)$, $\mathbf{A}_1 \in \mathcal{L}(E_1 \rightarrow E)$ where E_1 is linear normed space. Differential operators \mathbf{L} , \mathbf{L}_1 are as follows

$$\mathbf{L}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^n}{\partial t^n} + \sum_{k_0+k_1+\dots+k_m \leq n-1} a_{k_0\dots k_m}(x) \frac{\partial^{k_0+\dots+k_m}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_m^{k_m}},$$

$$\mathbf{L}_1\left(\frac{\partial}{\partial x}\right) = \sum_{k_0+k_1+\dots+k_m \leq n_1} b_{k_0\dots k_m}(x) \frac{\partial^{k_0+\dots+k_m}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_m^{k_m}}, \quad n_1 < n$$

where coefficients $a_{k_0\dots k_m} : \Omega \rightarrow \mathbb{R}^1$, $b_{k_0\dots k_m} : \Omega \rightarrow \mathbb{R}^1$ are sufficiently smooth and defined in $\overline{\Omega}$, $0 \in \Omega$. In the concrete cases the domains of definition of operators \mathbf{L}, \mathbf{L}_1 consist of linear manifolds E_∂ of sufficiently smooth abstract functions $u(x)$ with domain in Ω and their ranges in E , which satisfy certain system of homogeneous boundary conditions. The abstract function $f : \Omega \subset \mathbb{R}^{m+1} \rightarrow E$ of argument (t, x_1, \dots, x_m) is assumed to be given. It is requested to find the solution $u : \Omega \subset \mathbb{R}^{m+1} \rightarrow E_\partial$, abstract function $u(x)$ satisfy certain condition on boundary $\partial\Omega$. The concept of a skeleton chains is introduced as sequence of linear operators $\mathbf{B}_i \in \mathcal{L}(E_i \rightarrow E_i)$, $i = 1, 2, \dots, p$, where E_i are linear spaces corresponding to the skeleton decomposition of operator \mathbf{B} . It is assumed that irreversible operator \mathbf{B} generates skeleton chain of the finite length p . The problem is reduced to a regular split system with respect to higher order derivative terms with certain initial and boundary conditions.

Keywords: ill-posed problem, Cauchy problem, noninvertible operator, skeleton decomposition, PDE, BVP.

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Introduction

Let linear bounded operator \mathbf{B} acting from linear space E to E has no inverse operator. Differential operators

$$\mathbf{L}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^n}{\partial t^n} + \sum_{k_0+k_1+\dots+k_m \leq n-1} a_{k_0\dots k_m}(x) \frac{\partial^{k_0+\dots+k_m}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_m^{k_m}},$$

$$\mathbf{L}_1\left(\frac{\partial}{\partial x}\right) = \sum_{k_0+k_1+\dots+k_m \leq n_1} b_{k_0\dots k_m}(x) \frac{\partial^{k_0+\dots+k_m}}{\partial t^{k_0} \partial x_1^{k_1} \dots \partial x_m^{k_m}}, \quad n_1 < n$$

are defined. Here coefficients $a_{k_0\dots k_m} : \Omega \subset \mathbb{R}^{m+1} \rightarrow \mathbb{R}^1$, $b_{k_0\dots k_m} : \Omega \subset \mathbb{R}^{m+1} \rightarrow \mathbb{R}^1$ are sufficiently smooth and defined in $\overline{\Omega}$, $0 \in \Omega$. The domains of definition of operators \mathbf{L}, \mathbf{L}_1 consist of linear manifolds E_∂ sufficiently smooth functions in Ω with their ranges in E , which satisfy certain system of homogeneous boundary conditions.

Abstract function $f : \Omega \subset \mathbb{R}^{m+1} \rightarrow E$ of argument x is assumed to be given and the problem is to find the solution $u : \Omega \subset \mathbb{R}^{m+1} \rightarrow E_\partial$ which satisfy linear PDE

$$\mathbf{B}\mathbf{L}\left(\frac{\partial}{\partial x}\right)u = \mathbf{L}_1\left(\frac{\partial}{\partial x}\right)u + f(x). \quad (1)$$

Operator \mathbf{B} is assumed independent of x . If operator \mathbf{B} has inverse bounded operator then Eq. (1) is called *regular* and otherwise it is called *irregular* equation. If $E = \mathbb{R}^N$ and $\det B \neq 0$, then Eq. (1) is the system of linear partial differential equations (PDE) of Kovalevskaya type, and we have a well known regular problem of the PDE theory. The foundation of many branches of modern general theory of PDE systems was constructed by I.G. Petrovskii [1]. In regular case the initial conditions for Eq. (1) can be defined as follows

$$\left. \frac{\partial^i u}{\partial t^i} \right|_{t=0} = \varphi_i(x_1, \dots, x_m), \quad i = 0, 1, \dots, n-1. \quad (2)$$

Here functions φ_i are analytical functions in Ω . If f is analytic function in t, x_1, \dots, x_m in Ω , then Cauchy problem (1) – (2) is not only solvable but also *well-posed* in class of analytic functions.

The well-posedness of Cauchy problem is challenging issue even for linear PDE systems in spaces of non-analytic functions. They are usually solved in class of functions satisfying certain estimates [1]. As result of publication of S.L. Sobolev work [3] where he introduced nowadays called Sobolev equations, I.G. Petrovsky

and L.A. Lusternik on their seminars have attracted mathematicians attention to PDE systems unresolved with respect to the highest time derivative.

It is to be noted that there are numerous new models in the natural sciences, engineering, and mathematical economics (here readers may refer to [15–17] and other) formulated in terms of systems of Eqs 1.

Irregular models enable study of systems behavior in critical situations. At present, the basis of relevant theory is constructed for certain classes of equations. For example, the theory and numerical methods for differential-algebraic equations has been constructed.

The intensive studies of more complex theory of irregular PDE and abstract irregular differential operator equations are conducted but there are still a lot of unexplored problems.

If \mathbf{B} is normally solvable operator, $x \in \mathbb{R}^1$ then the approach in the theory of Eq. (1) can be based on the expansion of the Banach space into direct sum in accordance with Jordan structure of operator \mathbf{B} [4]- [10] and some results from the theory of semigroup with kernels [14]. These approaches are already employed for various problems of modern mathematical modeling, see e.g. [15–17].

In this field the analytical methods were proposed for constructing classical and generalized solutions of Cauchy problem for ordinary operator-differential equations for $x \in \mathbb{R}^1$ in Banach spaces with irreversible operator in the main part.

The theory of irregular operator-differential PDEs in Banach spaces in the multi-dimensional case for $x \in \mathbb{R}^n$, $n \geq 2$ to be constructed. There are only initial results in this field published in preprints [6, 7]. Therefore, the construction of the general theory of Eq. (1) with irreversible operator \mathbf{B} is of theoretical interest. It is also important for the modern mathematical models based on irregular systems.

It is to be outlined that classical initial Cauchy conditions (2) for Eq. (1) play very limited role. Indeed, because of the irreversibility the operator \mathbf{B} time direction is characteristic and functions φ_i can not be arbitrary selected in the initial conditions (2)! Then appear the question of reasonable formulation and methods of solution of non-classic boundary problems for the system (1), taking into account the structure of the operator \mathbf{B} . The objective of present work is to solve this problem. The similar Goursat problem was addressed in [8] using other methods. In sections 2 and 3 of present work this problem is solved for irreversible operator \mathbf{B} which enjoy skeleton decomposition $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, $\mathbf{A}_1 \in \mathcal{L}(E \rightarrow E_1)$, $\mathbf{A}_2 \in \mathcal{L}(E_1 \rightarrow E)$, where E_1 is normed space.

The remainder of the paper is organized as follows. Sec. 0.1 presents introductory

concerning the skeleton chains of linear operators using results [9]. The concept of a regular and singular skeleton chains is introduced. It is proved that operator \mathbf{B} must be nilpotent in case of singular skeleton chain. In Sec. 0.2 it is assumed that noninvertible operator \mathbf{B} generates a skeleton chain of linear operators of finite length p and it is demonstrated that irregular Eq. (1) can be reduced to the recurrent sequence of $p + 1$ equations. It is to be noted that each equation of this sequence is regular under the natural restrictions on differential operators \mathbf{L}, \mathbf{L}_1 and certain initial-boundary conditions. Therefore if operator \mathbf{B} has skeleton chain of length p then solution of irregular Eq. (1) can be reduced to regular system from $p + 1$ -th equation.

Proposed approach can be employed for wide range of concrete problems (1) due to finite length of skeleton chain of finite-dimensional operator \mathbf{B} .

The formulas relating the solution of Eq. (1) with the solution of reduced regular system are derived. This result allows us in Sec. 0.2 to set new well-posed non-classic boundary conditions for Eq. (1) for which the equation enjoy unique solution as demonstrated in Sec. 0.3 – 0.5. For applications, it is important that this solution can be found by solving the sequence of regular problem proposed in this paper. Corresponding results and examples are given in Sec. 0.3 and Sec. 0.4. Finally Sec. 0.5 demonstrates the results generalization to non-linear equations.

0.1 Skeleton chains of linear operator

Let $\mathbf{B} \in \mathcal{L}(E \rightarrow E)$, and $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, where $\mathbf{A}_2 \in \mathcal{L}(E \rightarrow E_1)$, $\mathbf{A}_1 \in \mathcal{L}(E_1 \rightarrow E)$, E_1, E are linear normed spaces. The following definitions can be introduced.

Decomposition $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$ is called *skeleton decomposition of operator \mathbf{B}* . Let us introduce linear operator $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1$. Obviously $\mathbf{B}_1 \in \mathcal{L}(E_1 \rightarrow E_1)$. If operator \mathbf{B}_1 has bounded inverse or it is null operator acting from E_1 to E_1 , then \mathbf{B} generate *skeleton chain* $\{\mathbf{B}_1\}$ of length 1. Then operator \mathbf{B}_1 can be called as *skeleton-attached operator* to operator \mathbf{B} . This chain is called *singular* if $\mathbf{B}_1 = 0$ and regular if $\mathbf{B}_1 \neq 0$. If \mathbf{B}_1 is irreversible non-null operator then it is assumed to have skeleton decomposition $\mathbf{B}_1 = \mathbf{A}_3 \mathbf{A}_4$, $\mathbf{A}_4 \in \mathcal{L}(E_1 \rightarrow E_2)$, $\mathbf{A}_3 \in \mathcal{L}(E_2 \rightarrow E_1)$, where E_2 is new linear normed space. Obviously in this case $\mathbf{A}_2 \mathbf{A}_1 = \mathbf{A}_3 \mathbf{A}_4$ and operator $\mathbf{B}_2 = \mathbf{A}_4 \mathbf{A}_3 \in \mathcal{L}(E_2 \rightarrow E_2)$ can be introduced. If it turns out that \mathbf{B}_2 has bounded inverse or $\mathbf{B}_2 \equiv 0$, then \mathbf{B} has skeleton chain $\{\mathbf{B}_1, \mathbf{B}_2\}$ of length 2. Chain $\{\mathbf{B}_1, \mathbf{B}_2\}$ is generate if $\mathbf{B}_2 = 0$ and nongenerate otherwise.

Therefore this process can be continued for a number of linear operators classes by introduction of the normed linear spaces $E_i, i = 1, \dots, p$ and by bounded

operators construction $\mathbf{A}_{2i} \in \mathcal{L}(E_{i-1} \rightarrow E_i)$, $\mathbf{A}_{2i-1} \in \mathcal{L}(E_i \rightarrow E_{i-1})$, which satisfy the following equalities

$$\mathbf{A}_{2i}\mathbf{A}_{2i-1} = \mathbf{A}_{2i+1}\mathbf{A}_{2i+2}, i = 1, 2, \dots, p-1. \quad (3)$$

Eq. (3) defines sequence of linear operators $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ as follows

$$\mathbf{B}_i = \mathbf{A}_{2i}\mathbf{A}_{2i-1}, i = 1, 2, \dots, p. \quad (4)$$

Obviously $\mathbf{B}_i \in \mathcal{L}(E_i \rightarrow E_i)$. Here operator \mathbf{B}_p either has inverse bounded or \mathbf{B}_p is null operator acting from E_p to E_p . This process can be formalized as following definition.

Definition 1.

Let $\mathbf{B} = \mathbf{A}_1\mathbf{A}_2$ and operators $\{\mathbf{A}_i\}_{i=1}^{2p}$ satisfy equality (3). Let operators $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ defined by formula (4), and operators $\{\mathbf{B}_1, \dots, \mathbf{B}_{p-1}\}$ are noninvertible, and operator \mathbf{B}_p has bounded inverse or null operators acting from E_p to E_p . Then operator \mathbf{B} generates skeleton chain of linear operators $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ on length p . If $\mathbf{B}_p \neq 0$ then the chain is *regular*, if $\mathbf{B}_p = 0$ then the chain is called *singular*. Operators $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ called skeleton-attached to operator \mathbf{B} .

The most important linear operators generating skeleton chains of the finite lengths are considered below:

1. Let $E = \mathbb{R}^n$, then square matrix $\mathbf{B} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\det \mathbf{B} = 0$ obviously has skeleton chain $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ of decreasing dimentionions. The final matrix \mathbf{B}_p will be regular or null matrix, $\det \mathbf{B}_i = 0$, $i = 1, \dots, p-1$.
2. Let E is infinite dimensional normed space, then finite operator $\mathbf{B} = \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i$, where $\{z_i\} \in E$, $\gamma_i \in E^*$ has skeleton chain consisting from finite number of matrices $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ of decreasing dimentionions. Here $\mathbf{B}_1 = ||\langle z_i, \gamma_j \rangle||_{i,j=1}^n$ is first element of this chain, $\det \mathbf{B}_i = 0$, $i = 1, \dots, p-1$. \mathbf{B}_p is null matrix or $\det \mathbf{B}_p \neq 0$.

Here according definition 1 length of the chain $p = 1$ if $\det[\langle z_i, \gamma_j \rangle]_{i,j=1}^n \neq 0$ or $\langle z_i, \gamma_j \rangle = 0, i, j = 1, 2, \dots, n$. In general case the chain is always consists from finite number of matrices.

Using (3), (4) and Definition 1 the following result can be formulated.

Lemma 1.

If operator \mathbf{B} has skeleton chain of length p then

$$\mathbf{B}^n = \mathbf{A}_1\mathbf{A}_3 \dots \mathbf{A}_{2n-1}\mathbf{B}_{n-1}\mathbf{A}_{2n-2}\mathbf{A}_{2n-4} \dots \mathbf{A}_2, n = 1, \dots, p+1, \quad (5)$$

where $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_p$ are elements of the skeleton chain of operator \mathbf{B} .

From Lemma 1 it follows

Corollary.

If operator \mathbf{B} has singular skeleton chain of length p , then \mathbf{B} is nilpotent operator of index $p + 1$.

To proof the Corollary it is sufficient to put $n = p + 1$ and demonstrate that degree \mathbf{B}^{p+1} is null operator because \mathbf{B}_p is null operator due to above introduced definition of singular skeleton chain.

0.2 Reduction of abstract irregular Eq. (1) to the sequence of regular equations

We always assume that operator \mathbf{B} and linear operators $\{\mathbf{A}_i\}_{i=1}^{2p}$ from skeleton chain of operator \mathbf{B} be independent on x and commutative with linear operators \mathbf{L} and \mathbf{L}_1 . In this paragraph for sake of clarity it is assumed that operators \mathbf{L} and \mathbf{L}_1 can be different from above introduced differential operators $\mathbf{L}(\frac{\partial}{\partial x})$, $\mathbf{L}_1(\frac{\partial}{\partial x})$ and equation can be considered in abstract form

$$\mathbf{B}\mathbf{L}u = \mathbf{L}_1u + f. \quad (6)$$

Eq. (1) can be considered as special case of Eq. (6). Obviously, the introduced commutativity condition is fulfilled for Eq. (1) with linear operator \mathbf{B} independent of x and introduced differential operators $\mathbf{L}(\frac{\partial}{\partial x})$, $\mathbf{L}_1(\frac{\partial}{\partial x})$.

Let us reduce Eq. (6) to system of $p + 1$ equations which are regular in certain conditions imposed on operators \mathbf{L}, \mathbf{L}_1 . Let us start with simple case $p = 1$. We introduce system of two equation

$$\mathbf{B}_1\mathbf{L}u_1 = \mathbf{L}_1u_1 + \mathbf{A}_2f, \quad (7)$$

$$\mathbf{L}_1u = -f + \mathbf{A}_1\mathbf{L}u_1. \quad (8)$$

where $u \in E$ and $u_1 \in E_1$. The decomposed system (7) – (8) can be obtained by formal multiplication of Eq. (6) on operator \mathbf{A}_2 from the skeleton decomposition of operator \mathbf{B} and making notation $u_1 = \mathbf{A}_2u$.

It is to be noted that system (7) – (8) is splitted and \mathbf{B}_1 is invertible operator. Therefore if operators $\mathbf{B}_1\mathbf{L} - \mathbf{L}_1$, and operator \mathbf{L}_1 have bounded inverse operators then the unique solution can be constructed. Of course without additional conditions there remains an open question: *is constructed solution $u(x)$ satisfy Eq. (6)?*

Let us introduce two lemmas establishing the link between Eq. (6) and system (7) – (8).

Lemma 2.

Let u^* satisfy Eq. (6) and operator \mathbf{L}_1 has left inverse. Then pair $u_1^* = \mathbf{A}_2 u^*, u^*$ satisfy system (7) – (8).

Proof. Based on conditions of the Lemma the following equality is satisfied

$$\mathbf{A}_1 \mathbf{A}_2 \mathbf{L} u^* = \mathbf{L}_1 u^* + f. \quad (9)$$

From (9) because of commutativity condition the following equality is valid

$$\mathbf{A}_1 \mathbf{L} \mathbf{A}_2 u^* = \mathbf{L}_1 u^* + f. \quad (10)$$

and

$$\mathbf{A}_2 \mathbf{A}_1 \mathbf{L} \mathbf{A}_2 u^* = \mathbf{L}_1 \mathbf{A}_2 u^* + \mathbf{A}_2 f. \quad (11)$$

The latter equality demonstrates that $u_1^* = \mathbf{A}_2 u^*$ is solution of Eq. (7). Substitution u_1^* into the right hand side of Eq. (8) yields the following equation with known right hand side

$$\mathbf{L}_1 u = -f + \mathbf{A}_1 \mathbf{A}_2 \mathbf{L} u^*$$

with respect to u . Here the operators commutativity property is employed. The solution exists for such equation. Indeed, due to Eq. (9) right hand side of the equation is equal to $\mathbf{L}_1 u^*$. Hence for given u^* and $u_1^* = \mathbf{A}_2 u^*$ the right hand side belongs to the range of operator \mathbf{L}_1 . Therefore, $\mathbf{L}_1 u = \mathbf{L}_1 u^*$. Because operator \mathbf{L}_1 has left inverse, then u^* is unique solution to Eq. (8) for $u_1 = \mathbf{A}_2 u^*$. Lemma 2 is proved. □

Lemma 3.

Suppose that there exists a pair (u_1^*, u^*) solution to Eq. (7), (8). Let operator \mathbf{L}_1 has right inverse \mathbf{L}_1^{-1} . Then element u^* satisfy Eq. (6).

Proof. Element $-f + \mathbf{A}_1 \mathbf{L} u_1^*$ belongs to the range of operator \mathbf{L}_1 , because pair u_1^*, u^* satisfies system of Eqs. (7) – (8). Hence $u^* = \mathbf{L}_1^{-1}(-f + \mathbf{A}_1 \mathbf{L} u_1^*)$ because $-f + \mathbf{A}_1 \mathbf{L} u_1^* \in \mathcal{R}(\mathbf{L}_1)$, where $\mathcal{R}(\mathbf{L}_1)$ is the range of operator \mathbf{L}_1 . It is to be demonstrated that constructed element u^* satisfy Eq. (6) because by hypothesis u_1^* satisfy Eq. (7). Indeed, substitution of the constructed u^* into Eq. (6), where $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, yields

$$\mathbf{A}_1 \mathbf{A}_2 \mathbf{L} \mathbf{L}_1^{-1}(-f + \mathbf{A}_1 \mathbf{L} u_1^*) = \mathbf{A}_1 \mathbf{L} u_1^*.$$

Taking into account operators commutativity the following equality is valid

$$\mathbf{A}_1 \mathbf{L} \{ \mathbf{A}_2 \mathbf{L}_1^{-1}(-f + \mathbf{A}_1 \mathbf{L} u_1^*) - u_1^* \} = 0 \quad (12)$$

Because \mathbf{A}_1 and \mathbf{L} are linear operators then it remains to verify in Eq. (12) element in braces is zero. Since $u_1^* \in E_1$ satisfy Eq. (7), where $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1$, then the following equality is valid

$$\mathbf{A}_2(\mathbf{A}_1 \mathbf{L} u_1^* - f) = \mathbf{L}_1 u_1^*. \quad (13)$$

Hence $\mathbf{A}_2(\mathbf{A}_1 \mathbf{L} u_1^* - f) \in \mathcal{R}(\mathbf{L}_1)$ and $u_1^* = \mathbf{L}_1^{-1} \mathbf{A}_2(\mathbf{A}_1 \mathbf{L} u_1^* - f)$, where \mathbf{L}_1^{-1} is right inverse to operator \mathbf{L}_1 . Because $\mathbf{A}_2 \mathbf{L}_1 = \mathbf{L}_1 \mathbf{A}_2$ then $\mathbf{A}_2 = \mathbf{L}_1^{-1} \mathbf{A}_2 \mathbf{L}_1$. This yields $\mathbf{A}_2 \mathbf{L}_1^{-1} = \mathbf{L}_1^{-1} \mathbf{A}_2 \mathbf{L}_1 \mathbf{L}_1^{-1}$. From equality $\mathbf{L}_1 \mathbf{L}_1^{-1} = \mathbf{I}$ it follows that right inverse \mathbf{L}_1^{-1} also commutative with operator \mathbf{A}_2 . Then Eq. (13) can be represented as

$$\mathbf{A}_2 \mathbf{L}_1^{-1}(\mathbf{A}_1 \mathbf{L} u_1^* - f) - u_1^* = 0.$$

Thus we have shown that expression in braces in Eq. (12) is zero. Lemma 3 is proved. \square

Let us concentrate on general case of skeleton chain of arbitrary finite length p . We assume operator \mathbf{B} to have skeleton chain $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$, $p \geq 1$, and linear operators $\{A_i\}_{i=1}^{2p}$ represent decomposition of \mathbf{B}_i which skeleton attached to \mathbf{B} . Introduce

$$u_i = \prod_{j=1}^i \mathbf{A}_{2j} u, \quad i = 1, \dots, p, \quad (14)$$

where $u_i \in E_i$, $u_i = \mathbf{A}_{2i} u_{i-1}$, $u_0 := u$.

If u_0 satisfy Eq. (6) then for $p \in \mathbb{N}$ by *Definition 1* we get equalities

$$\mathbf{B}_p \mathbf{L} u_p = \mathbf{L}_1 u_p + \prod_{j=1}^p \mathbf{A}_{2j} f, \quad (15)$$

$$\mathbf{L}_1 u_i = - \prod_{j=1}^i \mathbf{A}_{2j} f + \mathbf{A}_{2i+1} \mathbf{L} u_{i+1}, \quad (16)$$

$$\mathbf{L}_1 u = -f + \mathbf{A}_1 \mathbf{L} u_1. \quad (17)$$

For $p \geq 2$ there is connection between solution of Eq. (6) and system (15) – (17). In particular there are two lemmas.

Lemma 4.

Let u^* satisfy Eq. (6) and operator \mathbf{L}_1 has left inverse. Then elements $u_i^* = \prod_{j=1}^i \mathbf{A}_{2j} u^*$, $i = p, p-1, \dots, 1$ satisfy Eqs. (15), (16), and u^* satisfy Eq. (17).

Lemma 5.

Let elements $u_p^*, u_{p-1}^*, \dots, u_1^*, u^*$ satisfy Eqs (15), (16), (17) and operator \mathbf{L}_1 has right inverse. Then element u^* determined from Eq. (17) of split system (15), (16), (17) is the solution to Eq. (6).

Proof of Lemma 4 and Lemma 5 for any natural number p can be reduced to employment of the skeleton chain via operators $\{\mathbf{A}_j\}_{j=1}^{2p}$ and repeats stages of the proofs of Lemmas 2 and 3 for the case of $p = 1$.

Based on Lemmas 1–5 the following main result can be formulated.

Main Theorem. *Let noninvertible bounded operator \mathbf{B} has skeleton chain*

$$\{B_1, \dots, B_p\},$$

operator $\mathbf{B}_p \mathbf{L} - \mathbf{L}_1$ with definition domain in E_p has bounded inverse. Let operator \mathbf{L}_1 be defined on domains from E_i , $i = 1, \dots, p$ and E . Let operator \mathbf{L}_1 has inverse bounded operator. Then system (15), (16), (17) enjoy unique solution $\{u_p^, u_{p-1}^*, \dots, u_1^*, u^*\}$, defined as follows*

$$u_p^* = (\mathbf{B}_p \mathbf{L} - \mathbf{L}_1)^{-1} \prod_{j=1}^p \mathbf{A}_{2j} f,$$

$$u_i^* = \mathbf{L}_1^{-1} \left\{ - \prod_{j=1}^i \mathbf{A}_{2j} f + \mathbf{A}_{2j+1} \mathbf{L} u_{i+1}^* \right\}, i = p-1, \dots, 1,$$

$$u^* = \mathbf{L}_1^{-1} \{-f + \mathbf{A}_1 \mathbf{L} u_1^*\}.$$

Moreover, element u^* satisfy Eq. (6) and $u_i^* = \prod_{j=1}^i \mathbf{A}_{2j} u^*$, $i = 1, \dots, p$.

By setting the initial-boundary conditions to ensure the reversibility of the operators \mathbf{L}_1 and $\mathbf{B}_p \mathbf{L} - \mathbf{L}_1$ with specific differential operators \mathbf{L} and \mathbf{L}_1 and using the Main Theorem the existence and uniqueness theorems can be derived. Moreover, the formula obtained in Theorem can effectively build the desired classical solution

of Eq. (1) with sufficient smoothness of $f : \Omega \subset \mathbb{R}^{m+1} \rightarrow E$ and the coefficients of the differential operators \mathbf{L} and \mathbf{L}_1 . Such applications of the theory are discussed further in section 3.

0.3 The existence and methods of constructing solutions of nonclassic BVP with partial derivatives

0.3.1 System of Eqs. (18)

Consider the system

$$\mathbf{B} \sum_{k_1+k_2 \leq n} a_{k_1 k_2} \frac{\partial^{k_1+k_2} u(x, t)}{\partial t^{k_1} \partial x^{k_2}} = \sum_{k_1+k_2 \leq m} c_{k_1 k_2} \frac{\partial^{k_1+k_2} u(x, t)}{\partial t^{k_1} \partial x^{k_2}} + f(x, t). \quad (18)$$

Here $m < n$, \mathbf{B} is constant $N \times N$ matrix, $\det B = 0$, $a_{k_1 k_2}$, $c_{k_1 k_2}$ are numbers, $a_{n0} \neq 0$, $a_{0n} = 0$, $c_{0m} \neq 0$, $c_{m0} = 0$. Vector-functions $u(x, t) = (u_1(x, t), \dots, u_N(x, t))^T$, $f(x, t) = (f_1(x, t), \dots, f_N(x, t))^T$ are supposed to be defined and analytical for $-\infty < x, t < \infty$.

Let $\text{rank } \mathbf{B} = r < N$. Then based on [2] $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, where \mathbf{A}_1 is $N \times r$ matrix, \mathbf{A}_2 is $r \times N$ matrix. Let us introduce the $r \times r$ matrix $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1$ and assume $\det \mathbf{B}_1 \neq 0$. Then taking into account Lemma 3, the solution to system of Eqs (18) can be reduced to the successive solution of Eqs (7), (8), which are as follows in this case

$$\mathbf{B}_1 \sum_{k_1+k_2 \leq n} a_{k_1 k_2} \frac{\partial^{k_1+k_2} u_1(x, t)}{\partial t^{k_1} \partial x^{k_2}} = \sum_{k_1+k_2 \leq m} c_{k_1 k_2} \frac{\partial^{k_1+k_2} u_1(x, t)}{\partial t^{k_1} \partial x^{k_2}} + \mathbf{A}_2 f(x, t), \quad (19)$$

$$\sum_{k_1+k_2 \leq m} c_{k_1 k_2} \frac{\partial^{k_1+k_2} u(x, t)}{\partial t^{k_1} \partial x^{k_2}} = -f(x, t) + \mathbf{A}_1 \sum_{k_1+k_2 \leq n} a_{k_1 k_2} \frac{\partial^{k_1+k_2} u_1(x, t)}{\partial t^{k_1} \partial x^{k_2}}, \quad (20)$$

where $\det \mathbf{B}_1 \neq 0$, $u_1(x, t) = (u_{11}(x, t), \dots, u_{1r}(x, t))^T$, $r < N$, $u_1 = \mathbf{A}_2 u$. Since by hypothesis $a_{n0} \neq 0$, $c_{0m} \neq 0$, then for system (18) one may introduce the initial conditions

$$\left. \frac{\partial^i u(x, t)}{\partial x^i} \right|_{x=0}, \quad i = 0, 1, \dots, m-1, \quad (21)$$

$$\mathbf{A}_2 \frac{\partial^i u(x, t)}{\partial t^i} \Big|_{t=0} = 0, i = 0, 1, \dots, n-1. \quad (22)$$

Vector-function $u_1(x, t)$ based on Kovalevskaya theorem can be defined as unique solution to system (19) with initial conditions

$$\frac{\partial^i u_1(x, t)}{\partial t^i} \Big|_{t=0} = 0, i = 0, 1, \dots, n-1.$$

By vector $u_1(x, t)$ substitution into the right hand side of system (20), the desired vector $u(x, t)$ can be found as unique solution to the Cauchy problem (20) – (21).

0.3.2 System of Eqs. (23)

Consider system

$$\mathbf{B} \frac{\partial^n u(x, t)}{\partial x^n} = \left(\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \right) u(x, t) + f(x, t), n \geq 3. \quad (23)$$

As in system (18), \mathbf{B} is singular $N \times N$ matrix with $\text{rank } \mathbf{B} = r < N$, $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1$, $\det B_1 \neq 0$. Let $f(x, t) = (f_1(x, t), \dots, f_N(x, t))^T$ be vector-function defined for $0 \leq x \leq 1$, $0 < t < \infty$, continuous with respect to x and analytical by t , $u = (u_1, \dots, u_N)^T$.

The objective is to construct solution of system of Eqs (23) in $\Omega = \{0 \leq x \leq 1, 0 < t < \infty\}$. Based on Lemma 4 and Main Theorem introduce system of two equations ($u_1 = \mathbf{A}_2 u$)

$$\mathbf{B}_1 \frac{\partial^n u_1(x, t)}{\partial x^n} = \left(\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \right) u_1(x, t) + \mathbf{A}_2 f(x, t). \quad (24)$$

$$\left(\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \right) u(x, t) = -f(x, t) + \mathbf{A}_1 \frac{\partial^n u_1(x, t)}{\partial x^n}. \quad (25)$$

with initial-boundary conditions

$$\frac{\partial^i u_1(x, t)}{\partial x^i} \Big|_{x=0} = 0, i = 0, 1, \dots, n-1 \quad (26)$$

$$u(x, t)|_{t=0} = 0 \quad (27)$$

$$u(x, t)|_{x=0} = 0, u(x, t)|_{x=1} = 0. \quad (28)$$

Since $\det B_1 \neq 0$ then vector-function $u_1(x, t)$ based on Kovalevskaya theorem can be defined as unique solution of Cauchy problem (24) – (26). Substitute $u_1(x, t)$ in the right hand side of (25), the unique solution of the first boundary value problem (25), (27) (28) is constructed for heat equation using known formula (here readers may refer to p. 215 in [11]), using the source function. Constructed solution $u(x, t)$ will be classic unique solution of system (23) in domain $\Omega = \{0 \leq x \leq 1, 0 < t < \infty\}$. This solution satisfy initial conditions

$$\mathbf{A}_2 \frac{\partial^i u(x, t)}{\partial x^i} \Big|_{x=0} = 0, i = 1, \dots, n-1$$

and conditions (27) – (28).

0.3.3 Integro-differential equation of the first kind

Consider integro-differential equation of the first kind

$$\int_a^b K(x, s) \frac{\partial^3 u(s, y, t)}{\partial t^3} ds = \frac{\partial^2 u(x, y, t)}{\partial y^2} + au(x, y, t) + f(x, y, t), \quad (29)$$

where $x \in [a, b]$, $s \in [a, b]$, $y \in [0, 1]$, $0 \leq t < \infty$, $K(x, s) = \sum_{j=1}^n a_j(x)b_j(s)$ be continuous kernel, $f(x, y, t)$ is continuous function, $a \neq (\pi n)^2$. Here operator $\mathbf{B} := \int_a^b K(x, s)[\cdot] ds$, $\mathbf{L} = \frac{\partial^3}{\partial t^3}$, $\mathbf{L}_1 = \frac{\partial^2}{\partial y^2} + a$. Let

$$\det \left[\int_a^b a_i(s)b_j(s) ds \right]_{i,j=1}^n \neq 0.$$

Then using *Definition 1*, matrix $\mathbf{B}_1 = || \int_a^b a_j(s)b_i(s) ds ||_{i,j=1}^n$ be the regular skeleton chain of integral operator \mathbf{B} of length 1. Then using Lemma 3 the system (24)–(25) can be introduced in the following form

$$\mathbf{B}_1 \frac{\partial^3 u_1(y, t)}{\partial t^3} = \left(\frac{\partial^2}{\partial y^2} + a \right) u_1(y, t) + \beta(y, t), \quad (30)$$

where

$$\beta(y, t) = \left(\int_a^b f(x, y, t)b_1(x) dx, \dots, \int_a^b f(x, y, t)b_n(x) dx, \right)^T,$$

$$u_1(y, t) = (u_{11}(y, t), \dots, u_{1n}(y, t))^T,$$

$$u_{1j}(y, t) = \int_a^b b_j(s)u(s, y, t) ds, j = 1, 2, \dots, n, \det \mathbf{B}_1 \neq 0.$$

Then Eq. (29) has the next form

$$\frac{\partial^2 u(x, y, t)}{\partial y^2} + au(x, y, t) = -f(x, y, t) + \sum_{j=1}^n a_j(x) \frac{\partial^3}{\partial t^3} u_{1j}(y, t). \quad (31)$$

Following the Main Theorem the initial conditions can be defined as follows

$$\left. \frac{\partial^i u_1(y, t)}{\partial t^i} \right|_{t=0} = 0, i = 0, 1, 2 \quad (32)$$

and for Eq. (31) the boundary conditions can be defined as follows

$$u(x, y, t)|_{y=0} = 0, \quad (33)$$

$$u(x, y, t)|_{y=1} = 0. \quad (34)$$

Vector function $u_1(y, t)$ can be defined as solution of the regular Cauchy problem (30), (32). Substitute $u_1(y, t)$ into the right hand side of Eq. (31), where $a \neq (n\pi)^2$. Then desired solution $u(x, t)$ of Eq. (29) can be defined as unique solution of boundary problem (31), (33), (34) as follows

$$u(x, y, t) = \int_0^1 G(t, t_1) \left\{ -f(x, y, t_1) + \sum_{j=1}^n a_j(x) u_{1j}(y, t_1) \right\} dt_1,$$

where $G(t, t_1)$ is the Green function. Constructed solution $u(x, y, t)$ satisfies Eq. (29) and conditions

$$u(x, y, t)|_{y=0} = 0, u(x, y, t)|_{y=1} = 0,$$

$$\int_a^b b_j(s) \frac{\partial^i u(s, y, t)}{\partial t^i} ds \Big|_{t=0} = 0, j = 1, 2, \dots, n, i = 0, 1, 2.$$

0.4 Skeleton decomposition in the theory of irregular ODE in Banach space

Consider the simplest irregular ODE

$$\mathbf{B} \frac{du(t)}{dt} = u(t) + f(t), \quad (35)$$

$f(t) : [0, \infty) \rightarrow E$, $\mathbf{B} \in \mathcal{L}(E \rightarrow E)$. Let $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ is skeleton chain of operator \mathbf{B} . Then from Main Theorem the following next results can be formulated.

Theorem 1.

Let $\{\mathbf{B}_1, \dots, \mathbf{B}_p\}$ be regular skeleton chain, function $f(t)$ $p-1$ -times differentiable then Eq. (35) with initial condition

$$\prod_{j=1}^p A_{2j} u(t)|_{t=0} = c_0, \quad c_0 \in E_p \quad (36)$$

enjoy unique classic solution $u_0(t, c_0)$. Here

$$u_0(t, c_0) = -f(t) + \mathbf{A}_1 \frac{du_1}{dt}. \quad (37)$$

Let us outline the scheme for construction function $u_1(t, c_0)$ in solution (37) to problem (35) – (36) as follows:

1. If $p = 1$ then $u_1(t, c_0)$ satisfy the regular Cauchy problem

$$\begin{cases} \mathbf{B}_1 \frac{du_1}{dt} = u_1 + \mathbf{A}_2 f(t), \\ u_1(0) = c_0. \end{cases}$$

2. If $p \geq 2$ then function $u_1(t, c_0)$ can be constructed by following recursion

$$\begin{cases} \mathbf{B}_p \frac{du_p}{dt} = u_p + \prod_{j=1}^p \mathbf{A}_{2j} f(t), \\ u_p(0) = c_0. \end{cases}$$

$$u_i(t, c_0) = \mathbf{A}_{2i+1} \frac{du_{i+1}(t, c_0)}{dt} - \prod_{j=1}^i \mathbf{A}_{2j} f(t), \quad i = p-1, p-2, \dots, 1.$$

Theorem 2.

Let $\{\mathbf{B}_1, \dots, \mathbf{B}_{p-1}, 0\}$ be singular chain of length p , 0 is null operator acting from E_p to E_p be singular skeleton chain of length $p \geq 1$. Then \mathbf{B} be nilpotent operator and homogeneous eq. $\mathbf{B} \frac{du}{dt} = u$ has only trivial solution. In this case, if function $f(t)$ be p -times differentiable then unique classic solution of eq. (35) can be constructed as follows

$$u_n(t) = -f(t) + \mathbf{B} \frac{d}{dt} u_{n-1}(t), \quad u_0(t) = -f(t), \quad n = 1, 2, \dots, p.$$

Here function $u_p(t)$ be unique classic solution of eq. (35).

0.4.1 Examples

In order to illustrate Theorem 1 and Theorems 2 let us consider the following examples.

Example 1.

Let in Eq. (35)

$$\mathbf{B} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$f(t) = (f_1(t), f_2(t), f_3(t))^T$, $u(t) = (u_1(t), u_2(t), u_3(t))^T$. $\mathbf{B} = \mathbf{A}_1 \mathbf{A}_2$, where

$$\mathbf{A}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

$\mathbf{A}_2 = (a, \cdot)$ (scalar product), $a = (0, 1, 1)^T$. Since $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1 = 1$, then matrix \mathbf{B} generates the regular skeleton chain of length $p = 1$. Based on Theorem 2 the following system can be written

$$\begin{cases} \frac{dv}{dt} = v + f_2(t) + f_3(t), \\ v(0) = 0, \\ u(t) = -f(t) + \mathbf{A}_1 \frac{dv}{dt}, \end{cases}$$

where $v = \mathbf{A}_2 u = u_2(t) + u_3(t)$. The initial condition $u_2(0) + u_3(0) = 0$ has defined to meet condition $v(0) = 0$. It can be assumed that $f_i(t)$ is continuous function. Then

$$v(t) = \int_0^t e^{t-s} (f_2(s) + f_3(s)) ds$$

and system (35) is defined by matrix \mathbf{B} and initial condition $u_2(0) + u_3(0) = 0$ enjoy unique classic solution $u(t) = -f(t) + (2v(t) + 2f_2(t) + 2f_3(t), v(t) + f_2(t) + f_3(t), 0)^T$.

Example 2.

Let in Eq. (35)

$$\mathbf{B} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

$\mathbf{A}_2 = (a, \cdot)$, $a = (0, 1, -1)^T$. Then $\mathbf{B}_1 = \mathbf{A}_2 \mathbf{A}_1 = 0$, i.e. singular skeleton chain length $p = 1$ corresponds to matrix \mathbf{B} , \mathbf{B}^2 is null matrix. Therefore according to Theorem 2 solution to the system (35) with such matrix \mathbf{B} can be constructed as follows

$$u_n(t) = -f(t) + \mathbf{B} \frac{d}{dt} u_{n-1}(t), n = 1, 2, \dots, u_0(t) = -f(t).$$

For $n = 1$ we have the desired solution can be constructed as follows

$$u(t) = -f(t) + (2(f_2(t) - f_3(t))', (f_2(t) - f_3(t))', (f_2(t) - f_3(t))')^T.$$

If $f(t)$ is differentiable function, then we get classic solution. If $f_i(t)$ is piecewise absolutely continuous function with discontinuity points of the 1st kind and piecewise continuous derivative then we get solution in the space of distributions K' . Proposed method enables generalized solutions construction as well.

0.5 Generalizations and additional approaches to formulation of BVPs for the irregular PDEs

Suggested approach based on skeleton chain of operator \mathbf{B} can be employed for nonlinear equation as well. Indeed, let us replace in Eq. (1) $f(x)$ with nonlinear by u mapping $f(Mu, x)$. It is assumed $M = \prod_{j=1}^p \mathbf{A}_{2j}$, where $\{\mathbf{A}_{2j}\}$ are operators generate skeleton chain of operator \mathbf{B} . Then Eq. (15) of reduced system would be nonlinear equation in u_p . But in that case \mathbf{B}_p is invertible in Eq. (15). Therefore, it is easy to formulate analogues of Lemmas 1-5 and Main Theorem for such nonlinear case if mapping $f(Mu, x)$ is assumed Lipschitz in u .

The novel theory of non-classical boundary value problems for Eq. (1) can be developed taking into account the Jordan structure of operator \mathbf{B} . This approach has been used to study of boundary value problems for hyperbolic Goursat systems with singular matrix in the main part.

Using system (23) it is demonstrated below that the method of work [6] can also be applied to Eq. (1).

Indeed, let in system (23) matrix \mathbf{B} is $N \times N$ symmetric matrix, $\text{rank } \mathbf{B} = r$, and let $\{\varphi_i\}_1^{N-r}$ be orthonormal basis in $\ker \mathbf{B}$, $\varphi_i = (\varphi_{i1}, \dots, \varphi_{iN})^T$, $u = (u_1, \dots, u_N)^T$.

Introduce matrix

$$\Gamma = (\mathbf{B} + \sum_{i=1}^{N-r} (\cdot, \varphi_i) \varphi_i)^{-1}.$$

Then, without loss of generality, we can seek a solution of system (23) as following sum

$$u(x, t) = \Gamma v(x, t) + \sum_{i=1}^{N-r} c_i(x, t) \varphi_i, \quad (38)$$

where $v = (v_1, \dots, v_n)^T$,

$$(v(x, t), \varphi_i) = 0, \quad i = 1, 2, \dots, N - r. \quad (39)$$

Introduce projector $\mathbf{P} = \sum_{i=1}^{N-r} (\cdot, \varphi_i) \varphi_i$. Due to (39) $(\mathbf{I} - \mathbf{P})u = \Gamma v$, $\mathbf{B}\Gamma = \mathbf{I} - \mathbf{P}$. $\mathbf{P}v = 0$, $\mathbf{P}\Gamma = \Gamma\mathbf{P}$. Therefore, substitute (38) into system(23). Then application of operator $\mathbf{I} - \mathbf{P}$, followed by \mathbf{P} yield two equations

$$\frac{\partial^n v(x, t)}{\partial x^n} = \left(\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \right) \Gamma v(x, t) + (\mathbf{I} - \mathbf{P})f(x, t), \quad n \geq 3, \quad (40)$$

$$\left(\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \right) c(x, t) = \beta(x, t), \quad (41)$$

where $c(x, t) = (c_1(x, t), \dots, c_{N-r}(x, t))^T$, $\beta(x, t) = ((f(x, t), \varphi_1), \dots, (f(x, t), \varphi_{N-r}))^T$. Define for system (23) boundary conditions

$$(\mathbf{I} - \mathbf{P}) \frac{\partial^i u(x, t)}{\partial x^i} \Big|_{x=0} = 0, \quad i = 0, 1, \dots, n-1, \quad (42)$$

$$\mathbf{P}u(x, t)|_{t=0} = 0, \mathbf{P}u(x, t)|_{x=0} = 0, \mathbf{P}u(x, t)|_{x=1} = 0. \quad (43)$$

System (40) is Kovalevskaya type. Conditions (42) generate the following Cauchy conditions

$$\frac{\partial^i v(x, t)}{\partial x^i} \Big|_{x=0} = 0, \quad i = 0, 1, \dots, n-1 \quad (44)$$

for determination vector-function $v(x, t)$ from system (40).

Conditions (43) generate for parabolic Eqs. (41) the known boundary conditions

$$c_i(x, 0) = 0, c_i(0, t) = 0, c_i(1, t) = 0, \quad i = 1, 2, \dots, N-r. \quad (45)$$

If vector-function $f(x, t)$ is continuous on t and analytical on x , then $v(x, t)$ can be constructed as classic solution to Cauchy problem (40)–(44) based on Kovalevskaya theorem. Vector-function $c(x, t)$ can be defined by solution of parabolic Eq. (41) with conditions (45) in closed form using source function, see p. 215 in [11].

Other complicated boundary value problems for PDE systems can be raises and resolved using the generalized Jordan chains of linear operators. Recent results in the theory of generalized Jordan chains of linear operators are presented in [12].

Conclusion

This paper reports on the novel method of skeletal chains initiated in [9] for the linear operators in order to produce new non-classical boundary value problems for systems of differential and integral-differential equations with partial derivatives arising in the modern mathematical modeling.

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